The index of singular integral operators with discontinuous oscillating coefficients

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Abstract

The article presents the procedure of the index calculation for the elements of the algebra generated by one dimensional singular integral operators with discontinuous oscillating coefficients.

KEY WORDS: singular integral operator, oscillating coefficients, C^*-algebra, Fredholm operator, index, local representative

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In the work by I.B. Simonenko and I.Ts. Gohberg and N.Ja. Krupnik [1–4] the Fredholm theory of the singular integral operators with *piecewise continuous* coefficients was developed. This theory was based on the *local representatives* calculus for the operators considered. For the singular integral operators with discontinuous *oscillating* coefficients the local representatives calculus was obtained in [5]. In the present article we continue to investigate these objects and give a procedure of the index calculation for the operators under consideration. This procedure goes well for a finite set of points of oscillation and in order not to overload the presentation we describe it for the situation when the coefficients have the only one point of oscillation.

Let M be an oriented closed simple Lyapunov curve, B be the C^*-algebra generated by operators acting on $L^2(M)$ and having the form

$$b = c_1 + c_2 S,$$

where S is the singular integral operator and c_1 and c_2 are the operators of multiplication by functions from the space $C(M\backslash\{m_0\})$ having finite limits for $m \to m_0 \pm 0$, $m_0 \in M$.

The primitive ideal space Prim B of the algebra B can be represented as the disjoint union

$$\text{Prim } B = M_+ \cup M_- \cup \mathbf{R}_{m_0} \cup (m_0, +, \pm) \cup (m_0, -, \pm),$$

where M_+ and M_- are the copies of the set $M \setminus \{m_0\}$, $\mathbf{R}_{m_0}$ is the straight line, $(m_0, +, \pm)$, $(m_0, -, \pm)$ are four points. By $(m, \pm)$ we denote the points of $M_\pm$ respectively. The base of the topology on Prim B is defined in the following way: [6], [7, 33.12], [8, 53.12]

(a) a neighbourhoud of $(m_0, +, +)$ is the union of sets

$$\left[\bigcup_{m \in [m_0, m_1)} (m, +)\right] \cup \{t \in \mathbf{R}_{m_0} : t > N\},$$

where $N \in \mathbf{R}$ and m_1 is an arbitrary point such that $m_0 < m_1$ (here $<$ is the order defined by the orientation of M);

(b) a neighbourhood of $(m_0, -, +)$ is the union of sets

$$\left[\bigcup_{m \in (m_2, m_0]} (m, +)\right] \cup \{t \in \mathbf{R}_{m_0} : t < N\};$$

(c) a neighbourhood of $(m_0, -, -)$ is the union of sets

$$\left[\bigcup_{m \in [m_0, m_1)} (m, -)\right] \cup \{t \in \mathbf{R}_{m_0} : t > N\};$$

(d) a neighbourhood of $(m_0, +, -)$ is the union of sets

$$\left[\bigcup_{m \in (m_2, m_0]} (m, -)\right] \cup \{t \in \mathbf{R}_{m_0} : t < N\};$$

(e) a neighbourhood of $t \in \mathbf{R}_{m_0}$ is an open interval on $\mathbf{R}_{m_0}$ containing t;

(f) a neighbourhood of $(m, \pm)$ for $m \neq m_0$ is defined in the standard way (as a neighbourhood of a point on a curve).

As it has been already mentioned the local representatives calculus and the Fredholm theory of the operators from the algebra B was developed in [1 — 4].

The symbol of an element $b \in B$ is defined in terms of its local representatives that are given in the following way:

$$b^{\pm}(m) = c_1(m) \pm c_2(m), \ m \in M_{\pm},$$

$$b(m_0, +, \pm) = \lim_{m \to m_0 + 0} b^{\pm}(m) = c_1(m_0 + 0) \pm c_2(m_0 + 0),$$

$$b(m_0, -, \pm) = \lim_{m \to m_0 - 0} b^{\pm}(m) = c_1(m_0 - 0) \pm c_2(m_0 - 0),$$

$$b(m_0)(t) = \begin{pmatrix} \varphi_{11}(t) & \varphi_{12}(t) \\ \varphi_{21}(t) & \varphi_{22}(t) \end{pmatrix},$$

where $\varphi_{11}(t), \ \varphi_{22}(t) \in C(\overline{\mathbf{R}}), \ \varphi_{12}(t), \ \varphi_{21}(t) \in \overset{\circ}{C}(\overline{\mathbf{R}})$ and $C(\overline{\mathbf{R}})$ is the algebra of continuous functions having limits when $t \to \pm\infty$ and $\overset{\circ}{C}(\overline{\mathbf{R}})$ is the algebra of continuous functions tending to zero when $t \to \pm\infty$.

Moreover:

$$b(m_0)(+\infty) = \begin{pmatrix} b(m_0, +, +) & 0 \\ 0 & b(m_0, -, -) \end{pmatrix},$$

$$b(m_0)(-\infty) = \begin{pmatrix} b(m_0, +, -) & 0 \\ 0 & b(m_0, -, +) \end{pmatrix}.$$

It is known [1–4] that an operator $b \in B$ is Fredholm iff

(1) $b^{\pm}(m) \neq 0 \ \forall m \in M_{\pm}$;

(2) $\det [b(m_0)(t)] \neq 0 \ \forall t \in \mathbf{R}_{m_0}$;

(3) there exist nonzero limits

$$\lim_{t \to \pm\infty} \det [b(m_0)(t)] = \det [b(m_0)(\pm\infty)] .$$

And if (1)–(3) are satisfied then ([9], [10])

$$\operatorname{ind} b = -\frac{1}{2\pi} \left(\arg b^{+}(m)|_{M_+} - \arg b^{-}(m)|_{M_-} + \arg [\det b(m_0)(t)] \,|_{-\infty}^{+\infty} \right), \qquad (1)$$

where $[\ ]_{M\setminus\{m_0\}}$ is the increase of the function over the curve $M\setminus\{m_0\}$ and $[\ ]|_{t=-\infty}^{+\infty}$ is the increase of the function over the straight line $\mathbf{R}$.

Let $h\in\mathbf{R}$ and U_h be the operator of multiplication by a function $a_h(m)$ that is continuous on $M\setminus\{m_0\}$ and such that on a certain symmetric neighbourhood $O(m_0)$ of $m_0\in M$ $\quad a_h(m)$ has the form

$$a_h(m)=\begin{cases} e^{-\mathrm{i}h\ln(m_0-m)} & \text{where } m<m_0,\\ e^{-\mathrm{i}h\ln(m-m_0)} & \text{where } m_0<m,\end{cases} \tag{2}$$

and $a_h(m)$ is equal to 1 out of $O(m_0)$.

We denote by $C^*(B,U_h)$ the C^*-algebra generated by the algebra B and the operators $U_h,\ h\in\mathbf{R}$. The local representatives of elements of this algebra were constructed in [5] and in terms of the local representatives the conditions for the elements of $C^*(B,U_h)$ to be Fredholm operators were written out. Let us write out the local representatives for the operator U_h:

$$(U_h)^{\pm}(m)=a_h(m),\ m\in M_{\pm},$$

$$U_h(m_0,+,\pm)f(t)=U_h(m_0,-,\pm)f(t)=U_h(m_0)f(t)=T_hf(t),$$

where T_h is the shift operator $T_hf(t)=f(t+h),\quad f\in L^2(\mathbf{R},\mathbf{C}^2)$. In this case the matrix function $b(m_0)(t),t\in\mathbf{R}$ written out above is identified with the operator of multiplication by this matrix function in the space $L^2(\mathbf{R},\mathbf{C}^2)$.

The aim of this article is to calculate the index of a Fredholm operator $d\in C^*(B,U_h)$ that has the form

$$d=b_0+b_1U_h, \tag{3}$$

where $b_i\in B,\ i=0,1,\ h\in\mathbf{R}$ in terms of the elements of the algebra B, that is to give a procedure of finding an operator $d'\in B$, such that

$$\text{ind } d=\text{ind } d',$$

so that one can apply formula (1).

One of the principal technical instruments that are used in the procedure of the index calculation is a modification of theorem 46.20 [11], which can be written out in terms of the objects of the present paper in the following way.

Theorem 1. *The operator $d(m_0)=b_0(m_0)+b_1(m_0)T_h$ (that is the local representative of the operator d (3) on $\mathbf{R}_{m_0}$) in the space $L^2(\mathbf{R},\mathbf{C}^2)$ is invertible iff there exist non degenerate continuous matrix functions w_1 and s_1 (that are simultaneously diagonal at $\pm\infty$ or skew diagonal at one of the infinities and diagonal at the other infinity) such that*

$$w_1^{-1}d(m_0)s_1=e_0+e_1T_h, \tag{4}$$

where

$$e_0 = \begin{pmatrix} I_l & 0 \\ 0 & e^0_{22} \end{pmatrix}, \ e_1 = \begin{pmatrix} e^1_{11} & 0 \\ 0 & I_{2-l} \end{pmatrix}, \ l = 1, 2,$$

$$r(e^0_{22} T_h^{-1}) < 1, \quad r(e^1_{11} T_h) < 1,$$

or

$$w_1^{-1} d(m_0) s_1 = e_0 + e_1 T_h, \tag{5}$$

where

$$e_0 = \begin{pmatrix} e^0_{11} & 0 \\ 0 & I_{2-l} \end{pmatrix}, \ e_1 = \begin{pmatrix} I_l & 0 \\ 0 & e^1_{22} \end{pmatrix}, \ l = 1, 2,$$

$$r(e^0_{11} T_h) < 1, \quad r(e^1_{22} T_h^{-1}) < 1.$$

(The operators $e^0_{22} T_h^{-1}$ in (4) and $e^1_{22} T_h^{-1}$ in (5) are considered on the invariant subspace defined by the projection

$$\begin{pmatrix} 0 & 0 \\ 0 & I_{2-l} \end{pmatrix},$$

and the operators $e^0_{11} T_h$ in (4) and $e^1_{11} T_h$ in (5) are considered in the subspace defined by the projection

$$\begin{pmatrix} I_l & 0 \\ 0 & 0 \end{pmatrix},$$

here I_s is the identity operator acting on a space of dimension s.) Operator T_h is the shift operator:

$$T_h f(t) = f(t + h).$$

The proof of this theorem can be obtained by the same scheme as the proof of theorem 46.20 [11] by taking into account the explicit form of the operator $d(m_0)$.

Remark 1. The simultaneous diagonality (skew diagonality) of the matrices w_1 and s_1 at one and the same infinity follows from their construction (under the scheme of the proof of theorem 46.20 [11]) and from the fact that $b_i(m_0)(\pm\infty),\ i = 0, 1$ are the diagonal matrices.

Remark 2. We exclude from the considerations the situation when the matrices w_1 and s_1 are skew diagonal at both the infinities since in this case one can take the diagonal at $\pm\infty$ matrices

$$w_2(t) = w_1(t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad s_2(t) = s_1(t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

And in this way we obtain

$$w_2^{-1} d(m_0) s_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} w_1^{-1} d(m_0) s_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} =$$
$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (e_0 + e_1 T_h) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = e_0 + e_1 T_h.$$

A useful observation is the next

Lemma 1. *If the matrices w_1 and s_1 are simultaneously diagonal at $\pm\infty$ then there exist operators s and w from B that have zero indexes and are such that for their local representatives on $\mathbf{R}_{m_0}$ we have*

$$s(m_0) = s_1 \quad \textit{and} \quad w(m_0) = w_1.$$

Proof. By conditions of the lemma it is clear that the matrix functions s_1 and w_1 can be continued by means of non vanishing continuous functions $s^{\pm}$ and $w^{\pm}$ on $M_{\pm}$ in such a way that

$$\arg s^{+}(m)|_{M_+} - \arg s^{-}(m)|_{M_-} = \arg\left[\det s_1(t)\right]|_{-\infty}^{+\infty},$$

$$\arg w^{+}(m)|_{M_+} - \arg w^{-}(m)|_{M_-} = \arg\left[\det w_1(t)\right]|_{-\infty}^{+\infty}.$$

After this is done it is enough to apply formula (1). $\square$

We start our considerations with the following technical statement.

Lemma 2. *Let $A(t), B(t), X(t), Y(t), Z_0(t), Z_1(t),\ t \in \overline{\mathbf{R}}_{m_0}$ be two dimensional continuous matrix functions such that $X(t), Y(t)$ are non degenerate at every t and $A(\pm\infty)$ and $B(\pm\infty)$ are diagonal.*

Let also ∞ denotes one of the (fixed) infinities: either $+\infty$ or $-\infty$ and $A(\infty) = [a_{ij}]_{i,j=1}^2$, $B(\infty) = [b_{ij}]_{i,j=1}^2$ and so on.

If in the space $L^2(\mathbf{R}, \mathbf{C}^2)$ the operators of multiplication by matrix functions A, B, X, Y, Z_0, Z_1 satisfy the equality

$$X(A + BT_h)Y = Z_0 + Z_1 T_h, \tag{6}$$

then:

(i) the matrices $Z_0(t)$ and $Z_1(t)$ have the form

$$Z_0(t) = \begin{pmatrix} z_{11}^0(t) & 0 \\ 0 & 1 \end{pmatrix}, \quad Z_1(t) = \begin{pmatrix} 1 & 0 \\ 0 & z_{22}^1(t) \end{pmatrix},$$

where $|z^0_{11}| < 1$ *and* $|z^1_{22}| < 1$ *iff*
$X(\infty)$ *and* $Y(\infty)$ *are simultaneously diagonal and*

$$|a_{11}| < |b_{11}|, \ |a_{22}| > |b_{22}|, \ a_{22}x_{22}y_{22} = 1, \ b_{11}x_{11}y_{11} = 1 \tag{7}$$

or

$X(\infty)$ *and* $Y(\infty)$ *are simultaneously skew diagonal and*

$$|a_{11}| > |b_{11}|, \ |a_{22}| < |b_{22}|, \ a_{11}x_{21}y_{12} = 1, \ b_{22}x_{12}y_{21} = 1; \tag{8}$$

(ii) the matrices $Z_0(t)$ *and* $Z_1(t)$ *have the form*

$$Z_0(t) = \begin{pmatrix} 1 & 0 \\ 0 & z^0_{22}(t) \end{pmatrix}, \quad Z_1(t) = \begin{pmatrix} z^1_{11}(t) & 0 \\ 0 & 1 \end{pmatrix},$$

where $|z^0_{22}| < 1$ *and* $|z^1_{11}| < 1$ *iff*
$X(\infty)$ *and* $Y(\infty)$ *are simultaneously diagonal and*

$$|a_{11}| > |b_{11}|, \ |a_{22}| < |b_{22}|, \ a_{11}x_{11}y_{11} = 1, \ b_{22}x_{22}y_{22} = 1 \tag{9}$$

or

$X(\infty)$ *and* $Y(\infty)$ *are simultaneously skew diagonal and*

$$|a_{11}| < |b_{11}|, \ |a_{22}| > |b_{22}|, \ a_{22}x_{12}y_{21} = 1, \ b_{11}x_{21}y_{12} = 1; \tag{10}$$

(iii) if

$$Z_0(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Z_1 = \begin{pmatrix} z^1_{11}(t) & 0 \\ 0 & z^1_{22}(t) \end{pmatrix},$$

where $|z^1_{11}| < 1$ *and* $|z^1_{22}| < 1$ *at least for one infinity then* $X(\pm\infty)$ *and* $Y(\pm\infty)$ *can be taken to be diagonal at both the infinities and*

$$|a_{11}| > |b_{11}|, \ |a_{22}| > |b_{22}|, \ a_{11}x_{11}y_{11} = 1, \ a_{22}x_{22}y_{22} = 1; \tag{11}$$

(iv) if

$$Z_0(t) = \begin{pmatrix} z^0_{11}(t) & 0 \\ 0 & z^0_{22}(t) \end{pmatrix}, \quad Z_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $|z^0_{11}| < 1$ *and* $|z^0_{22}| < 1$ *at least at one infinity then* $X(\pm\infty)$ *and* $Y(\pm\infty)$ *can be taken to be diagonal at both the infinities and*

$$|a_{11}| > |b_{11}|, \ |a_{22}| > |b_{22}|, \ b_{11}x_{11}y_{11} = 1, \ b_{22}x_{22}y_{22} = 1. \tag{12}$$

Proof. Under the conditions of (i) the equality (6) is equivalent to the system:

$$\begin{cases} b_{11}x_{11}y_{11} + b_{22}x_{12}y_{21} = 1, \\ a_{11}x_{11}y_{12} + a_{22}x_{12}y_{22} = 0, \\ b_{11}x_{11}y_{12} + b_{22}x_{12}y_{22} = 0, \\ a_{11}x_{21}y_{11} + a_{22}x_{22}y_{21} = 0, \\ b_{11}x_{21}y_{11} + b_{22}x_{22}y_{21} = 0, \\ a_{11}x_{21}y_{12} + a_{22}x_{22}y_{22} = 1, \\ |a_{11}x_{11}y_{11} + a_{22}x_{12}y_{21}| < 1, \\ |b_{11}x_{21}y_{12} + b_{22}x_{22}y_{22}| < 1. \end{cases} \tag{13}$$

By solving the second and the third equations of the system with respect to the unknowns $x_{11}y_{12}$ and $x_{12}y_{22}$ we obtain that if $a_{11}b_{22} - a_{22}b_{11} \neq 0$ then

$$x_{11}y_{12} = x_{12}y_{22} = 0.$$

Under the same condition the forth and the fifth equations imply

$$x_{21}y_{11} = x_{22}y_{21} = 0.$$

If $x_{11} = 0$ then $x_{12} \neq 0$, $x_{21} \neq 0$ (since the matrix $X(\infty)$ is non degenerate), so $y_{11} = y_{22} = 0$. Therefore $y_{12} \neq 0$, $y_{21} \neq 0$ (since the matrix $Y(\infty)$ is non degenerate) and $x_{22} = 0$. Thus we have shown that the matrices $X(\infty)$ and $Y(\infty)$ are skew diagonal.

Substituting $x_{11} = x_{22} = y_{11} = y_{22} = 0$ into the first and the sixth equations and the inequalities of system (13) we obtain (8).

If $x_{11} \neq 0$ then one can show in an analogous way that $X(\infty)$ and $Y(\infty)$ are diagonal and (7) holds.

If $\frac{a_{11}}{a_{22}} = \frac{b_{11}}{b_{22}} = \lambda \neq 0$, then it follows from (13) that

$$\begin{cases} |a_{22}||\lambda x_{11}y_{11} + x_{12}y_{21}| < 1, \\ b_{22}(\lambda x_{11}y_{11} + x_{12}y_{21}) = 1, \\ |b_{22}||\lambda x_{21}y_{12} + x_{22}y_{22}| < 1, \\ a_{22}(\lambda x_{21}y_{12} + x_{22}y_{22}) = 1. \end{cases}$$

And we arrive at the inequalities

$$\left|\frac{a_{22}}{b_{22}}\right| < 1 \quad \text{and} \quad \left|\frac{b_{22}}{a_{22}}\right| < 1$$

which is a contradiction.

The case (ii) can be considered in an analogous way. The corresponding system has the form:

$$\begin{cases} a_{11}x_{11}y_{11} + a_{22}x_{12}y_{21} = 1, \\ a_{11}x_{11}y_{12} + a_{22}x_{12}y_{22} = 0, \\ b_{11}x_{11}y_{12} + b_{22}x_{12}y_{22} = 0, \\ a_{11}x_{21}y_{11} + a_{22}x_{22}y_{21} = 0, \\ b_{11}x_{21}y_{11} + b_{22}x_{22}y_{21} = 0, \\ b_{11}x_{21}y_{12} + b_{22}x_{22}y_{22} = 1, \\ |b_{11}x_{11}y_{11} + b_{22}x_{12}y_{21}| < 1, \\ |a_{11}x_{21}y_{12} + a_{22}x_{22}y_{22}| < 1. \end{cases}$$

In the case (iii) we obtain the non degeneracy of the matrix $A(t)$ for every $t \in \overline{\mathbf{R}}_{m_0}$ and the matrix $X(t)$ can be chosen to be equal to $A^{-1}(t)$ and $Y(t)$ to be equal to the identity matrix. Thus (11) holds true.

The case (iv) can be proved in the similar way. The proof is complete. □

The properties of the topology of Prim B imply that not changing the index of a Fredholm operator d of the form (3) one can change its coefficients and operators w and s in such a way that $b_i(m)$, $w(m)$, $s(m)$ will be constant (equal respectively to $b_i(m_0 \pm 0)$, $w(m_0 \pm 0)$, $s(m_0 \pm 0)$) in a certain symmetric neighbourhood $O'(m_0) \subset O(m_0)$. In what follows we shall consider precisely such 'changed' operators.

When calculating the index of d we shall consider in sequel all the possible situations mentioned in theorem 1.

Theorem 2. *A Fredholm operator d of the form (3) satisfies the inequalities (case I)*

$$|b_0^{\pm}(m_0+0)| > |b_1^{\pm}(m_0+0)|, \;\; |b_0^{\pm}(m_0-0)| > |b_1^{\pm}(m_0-0)| \tag{14}$$

or the inequalities (case II)

$$|b_0^{\pm}(m_0+0)| < |b_1^{\pm}(m_0+0)|, \;\; |b_0^{\pm}(m_0-0)| < |b_1^{\pm}(m_0-0)|) \tag{15}$$

iff there exist zero index operators w and s from the algebra B such that the local representative for $e = w^{-1}ds$ on the straight line $\mathbf{R}_{m_0}$ is the operator (4) (respectively (5)) for $l = 2$. In this case there exists a homotopy in the class of Fredholm operators of $C^*(B, U_h)$ of the operator e to an operator $d' \in B$ and therefore

$$\text{ind } d = \text{ind } e = \text{ind } d'.$$

Proof. The proof exploits theorem 1 and we shall use the same notation. The inequalities (14) and (15) were obtained in lemma 2 for the cases (iii) and (iv) respectively ($X(t) = w_1^{-1}(t)$, $Y(t) = s_1(t)$, $A(t) = b_0(m_0)(t)$, $B(t) = b_1(m_0)(t)$, $Z_0(t) =$

$e_0(t),\ Z_1(t) = e_1(t))$.

By lemma 1 there exist zero index operators w and s such that $w(m_0)(t) = w_1(t)$ and $s(m_0)(t) = s_1(t)$.

In case I it follows from (11) that

$$e^{\pm}(m_0+0) = 1 + \frac{b_1^{\pm}(m_0+0)}{b_0^{\pm}(m_0+0)}T_h,$$

$$e^{\pm}(m_0-0) = 1 + \frac{b_1^{\pm}(m_0-0)}{b_0^{\pm}(m_0-0)}T_h.$$

Now we shall construct the homotopy of the operator e to an operator $e' \in B$.

To start with we define the homotopy of the operator $e(m_0) = e_0(m_0) + e_1(m_0)T_h$ (that is the homotopy of the corresponding local representatives) on $\mathbf{R_{m_0}} \cup (m_0,+,\pm) \cup (m_0,-,\pm)$.

For every $\tau \in [0,1]$ we set

$$(e'_\tau)(m_0)(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} (1-\tau)e^1_{11}(t) & 0 \\ 0 & (1-\tau)e^1_{22}(t) \end{pmatrix} T_h,$$

$$e'_\tau(m_0,+,\pm) = 1 + (1-\tau)\frac{b_1^{\pm}(m_0+0)}{b_0^{\pm}(m_0+0)}T_h,$$

$$e'_\tau(m_0,-,\pm) = 1 + (1-\tau)\frac{b_1^{\pm}(m_0-0)}{b_0^{\pm}(m_0-0)}T_h.$$

For $\tau = 1$ we have

$$e'_1(m_0)(t) = e_0(m_0)(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$e'_1(m_0,+,\pm) = e'_1(m_0,-,\pm) = 1.$$

Now let us extend this homotopy onto M_+ and M_-:

$$(e'_\tau)^{\pm}(m) = (w^{-1})^{\pm}(m)(b_0^{\pm}(m) + \varphi_\tau(m)b_1^{\pm}(m)a_h(m))s^{\pm}(m),$$

where

$$\varphi_\tau(m) = \begin{cases} 1 - \tau \cdot \frac{m-m_1}{m_0-m_1} & \text{when } m \in (m_0,m_1), \\ 1 - \tau \cdot \frac{m-m_2}{m_0-m_2} & \text{where } m \in (m_2,m_0), \\ 1 & \text{where } m \notin (m_2,m_0) \cup (m_0,m_1), \end{cases}$$

and m_1 is a certain point from the neighbourhood on M_+ and M_-neighbourhood on M_+ and M_- corresponding to the neighbourhood $O'(m_0)$ on $M \setminus \{m_0\}$ and m_2 is the point that is symmetric to m_1 with respect to m_0.

The operators $(e'_\tau)^\pm(m)$ are invertible for every $\tau \in [0,1]$ since in the opposite case in the neighbourhood corresponding to $O'(m_0)$ one obtains

$$b_0^\pm(m) + \varphi_\tau(m) b_1^\pm(m) a_h(m) = 0.$$

And it follows that

$$|b_0^\pm(m)| = |\varphi_\tau(m)||b_1^\pm(m)|.$$

But since the functions $b_i^\pm(m),\ i = 1,2,$ are constants on this neighbourhood and $|\varphi_\tau(m)| \leq 1$ this leads to a contradiction with (14).

For every $\tau \in [0,1]$ the invertible operators $(e'_\tau)^\pm(m)$, $e'_\tau(m_0,+,\pm)$, $e'_\tau(m_0,-,\pm)$ and $e'_\tau(m_0)$ define the symbol of a certain Fredholm operator $e'_\tau \in C^*(B, U_h)$ (they are its local representatives). Thus

$$\text{ind } d = \text{ind } e = \text{ind } e'_0 = \text{ind } e'_1,$$

but e'_1 is an element of the algebra B. Set $d' = e'_1$. The index of the latter operator can be calculated by means of (1).

In order to reduce case II to case I it is enough to multiply the operator d by U_h^{-1} (under this operation index does not change). The theorem is proved. $\square$

To prove the next theorem we need the operator $P_1 + Q_1 U_h^{-1}$ where P_1 and Q_1 are the operators of multiplication by the functions p_1 and q_1 having in $O'(m_0)$ the form respectively

$$p_1(m) = \begin{cases} 0 & \text{when } m < m_0, \\ 1 & \text{when } m_0 < m, \end{cases} \qquad q_1(m) = \begin{cases} 1 & \text{when } m < m_0, \\ 0 & \text{when } m_0 < m, \end{cases}$$

and such that $p_1(m) + q_1(m)a_{-h}(m) \neq 0$ out of $O'(m_0)$.

Let us write out the local representatives for this operator:

$$(P_1 + Q_1 U_{-h})^\pm = \begin{cases} a_{-h} & \text{when } m < m_0,\ m \in O'(m_0), \\ 1 & \text{when } m_0 < m,\ m \in O'(m_0), \\ p_1(m) + q_1(m)a_{-h}(m) & \text{when } m \notin O'(m_0), \end{cases}$$

$$(P_1 + Q_1 U_{-h})(m_0,+,\pm) = 1, \quad (P_1 + Q_1 U_{-h})(m_0,-,\pm) = T_{-h},$$

$$(P_1 + Q_1 U_{-h})(m_0)(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} T_{-h}.$$

Theorem 3. *A Fredholm operator d of the form (3) satisfies the inequalities (case III)*

$$|b_0^{\pm}(m_0+0)| > |b_1^{\pm}(m_0+0)|, \;\; |b_0^{\pm}(m_0-0)| < |b_1^{\pm}(m_0-0)| \tag{16}$$

or the inequalities (case IV)

$$|b_0^{\pm}(m_0+0)| < |b_1^{\pm}(m_0+0)|, \;\; |b_0^{\pm}(m_0-0)| > |b_1^{\pm}(m_0-0)|) \tag{17}$$

iff there exist zero index operators w and s from B such that the local representative for $e = w^{-1}ds$ on $\mathbf{R}_{m_0}$ is equal to (4) (is equal to (5)) for $l = 1$. Moreover there exists a homotopy in the class of Fredholm operators from $C^*(B, U_h)$ of the operator e to the operator e' such that in case III

$$d' = e'(P_1 + Q_1 U_{-h}) \in B$$

and in case IV

$$d' = e'(P_1 U_h + Q_1) \in B$$

and

$$\text{ind } d = \text{ind } d'.$$

Proof. The proof of this theorem is also based on theorem 1 and thus we are using the same notation. The inequalities (16) and (17) follow from the cases (i) and (ii) of lemma 2 where one should take the corresponding infinities. The further proof is analogous to that of the preceding theorem.

In case III for every $\tau \in [0, 1]$ we set

$$(e'_\tau)^{\pm}(m) = \begin{cases} (w^{-1})^{\pm}(m)(\varphi_\tau(m) b_0^{\pm}(m) + b_1^{\pm}(m) a_h(m)) s^{\pm}(m) & \text{when } m < m_0, \\ (w^{-1})^{\pm}(m)(b_0^{\pm}(m) + \varphi_\tau(m) b_1^{\pm}(m) a_h(m)) s^{\pm}(m) & \text{when } m_0 < m, \end{cases}$$

$$e'_\tau(m_0, +, \pm) = 1 + (1-\tau)\frac{b_1^{\pm}(m_0+0)}{b_0^{\pm}(m_0+0)} T_h,$$

$$e'_\tau(m_0, -, \pm) = (1-\tau)\frac{b_0^{\pm}(m_0-0)}{b_1^{\pm}(m_0-0)} + T_h,$$

$$e'_\tau(m_0)(t) = \begin{pmatrix} 1 & 0 \\ 0 & (1-\tau) e_{22}^0(t) \end{pmatrix} + \begin{pmatrix} (1-\tau) e_{11}^1(t) & 0 \\ 0 & 1 \end{pmatrix} T_h,$$

where the function φ_τ is defined in the same way as in the proof of the preceding theorem.

For $\tau = 1$ we have

$$(e'_1)^{\pm}(m) = \begin{cases} (w^{-1})^{\pm}(m)(\varphi_1(m) b_0^{\pm}(m) + b_1^{\pm}(m) a_h(m)) s^{\pm}(m) & \text{when } m < m_0, \\ (w^{-1})^{\pm}(m)(b_0^{\pm}(m) + \varphi_1(m) b_1^{\pm}(m) a_h(m)) s^{\pm}(m) & \text{when } m_0 < m, \end{cases}$$

$$e'_1(m_0,+,\pm)=1,\quad e'_1(m_0,-,\pm)=T_h,$$

$$e'_1(m_0)(t)=\begin{pmatrix}1&0\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\0&1\end{pmatrix}T_h.$$

For every $\tau\in[0,1]$ the invertible operators $(e'_\tau)^{\pm}(m)$, $e'_\tau(m_0,+,\pm)$, $e'_\tau(m_0,-,\pm)$ and $e'_\tau(m_0)$ define the symbol of a certain Fredholm operator e'_τ and

$$\text{ind } d=\text{ind } e=\text{ind } e'_0=\text{ind } e'_1.$$

Set $e'=e'_1$. Clearly the operator $d'=e'(P_1+Q_1U_{-h})$ belongs to the algebra and its index can be calculated by (1). Thus

$$\text{ind } d=\text{ind } e'=\text{ind } (e'(P_1+Q_1U_{-h})).$$

To reduce the case IV to the case III it is enough to multiply the operator d by U_h^{-1} . The proof is complete. □

An important technical step in the investigation of the cases to be considered is the next result.

Theorem 4. *Let P and Q be the projections on $L^2(M)$ having the form*

$$P(m)=\frac{1}{2}(I+S),\quad Q(m)=\frac{1}{2}(I-S),$$

where I is the identity operator and S is the singular integral operator. Then the operators $P+QU_h$ and $Q+PU_h$ are Fredholm and their indexes are zero.

Proof. Since $Q+PU_h=(P+QU_{-h})U_h$ it is enough to verify the statement for the operator $P+QU_h$.

An operator is Fredholm iff all its local representatives are invertible. The explicit form of the local representatives for $P+QU_h$ is written out below and it is clear that all of them are invertible:

$$((P+QU_h)^+)^{-1}(m)=1,\quad ((P+QU_h)^-)^{-1}(m)=a_{-h}(m),$$

$$((P+QU_h)(m_0,\pm,+)^{-1}=1,\quad ((P+QU_h)(m_0,\pm,-)^{-1}=T_{-h},$$

$$((P+QU_h)(m_0)(t))^{-1}=$$

$$=\frac{1}{e^{\pi h}+e^{2\pi t}}\left(\begin{pmatrix}e^{2\pi t}&\mathrm{i}e^{\pi t}\\-\mathrm{i}e^{\pi t+\pi h}&e^{\pi h}\end{pmatrix}+\begin{pmatrix}e^{\pi h}&-\mathrm{i}e^{\pi t}\\\mathrm{i}e^{\pi t+\pi h}&e^{2\pi t}\end{pmatrix}T_{-h}\right).$$

Let us show now that the index of $P+QU_h$ is zero. Let $F_\alpha:L^2(M)\to L^2(M)$ be the operator of the form

$$\big[F_\alpha(f)\big](m)=|\alpha'(m)|^{1/2}f(\alpha(m)),$$

where the diffeomorphism $\alpha : M \to M$ changes the orientation of the curve M and on the neighbourhood $O(m_0)$ acts as the reflection and $\alpha(m_0) = m_0,\ \alpha^2(m) = m$. It is known (see [12]) that this operator possesses the following properties:

(1) $F_\alpha^2 = I$,

(2) $F_\alpha P F_\alpha \sim Q$,

(3) $F_\alpha Q F_\alpha \sim P$,

(4) $F_\alpha U_h F_\alpha = U_h$

(here the sign $\sim$ means the the left hand part and the right hand part differ by a compact summand).

It follows from (2)—(4) that

$$F_\alpha(P + QU_h)F_\alpha = F_\alpha P F_\alpha + F_\alpha Q F_\alpha F_\alpha U_h F_\alpha \sim Q + PU_h.$$

Thus

$$\text{ind } (P + QU_h) = \text{ind } (Q + PU_h) = \text{ind } (PU_h^* + Q) = \text{ind } (QU_h^* + P),$$

and therefore

$$2\text{ind } (P + QU_h) = \text{ind } \big[(P + QU_h)(P + QU_h^*)\big] = \text{ind } \big[P + QU_hP + QU_hQU_h^*\big],$$

where $\tau \in [0, 1]$.

Consider the operator

$$X + (1 - \tau)YU_h = \big[P + QU_hQU_h^*\big] + (1 - \tau)QU_hP.$$

Since

$$(X + (1 - \tau)YU_h)^{\pm}(m) = 1,$$
$$(X + (1 - \tau)YU_h)(m_0, +, \pm) = (X + (1 - \tau)YU_h)(m_0, -, \pm) = 1,$$
$$\det\ (P + QU_hQU_h^*)(m_0)(t) = \frac{(1 + e^{\pi h + 2\pi t})^2}{(1 + e^{2\pi t})(1 + e^{2\pi h + 2\pi t})} \neq 0 \quad \forall t \in \mathbf{R}_{m_0},$$
$$X(m_0)(\pm\infty) = 1 > 0 = Y(m_0)(\pm\infty).$$

it follows from theorem 2 that the operator $X + (1 - \tau)YU_h$ is Fredholm.

Thus

$$2\text{ind } (P + QU_h) =$$
$$= \text{ind } \big[P + (1 - \tau)QU_hP + QU_hQU_h^*\big] = \text{ind } \big[P + 0 \cdot QU_hP + QU_hQU_h^*\big] =$$
$$= -\frac{1}{2\pi}\left(\arg 1|_{M_+} - \arg 1|_{M_-} + \arg\left[\frac{(1 + e^{\pi h + 2\pi t})^2}{(1 + e^{2\pi t})(1 + e^{2\pi h + 2\pi t})}\right]\Bigg|_{t=-\infty}^{+\infty}\right) = 0.$$

The theorem is proved. $\square$

Remark 3. The operator $(P+QU_h)(m_0)(t)$ satisfies the equality

$$\widetilde{w}_1^{-1}(t)\Big[(P+QU_h)(m_0)(t)\Big]\widetilde{s}_1(t)=e_0(t)+e_1(t)T_h$$

where

$$e_0(t)=\begin{pmatrix}0&0\\0&1\end{pmatrix},\quad e_1(t)=\begin{pmatrix}1&0\\0&0\end{pmatrix},$$

$$\widetilde{w}_1(t)=\begin{pmatrix}1&-\mathrm{i}e^{\pi t}\\ \mathrm{i}e^{\pi t}&1\end{pmatrix},$$

$$\widetilde{s}_1(t)=\begin{pmatrix}\frac{e^{-\pi h}(1+e^{\pi t})(e^{2\pi h}+e^{2\pi t})}{(1+e^{\pi t})(e^{\pi h}+e^{2\pi t})(1+e^{\pi t-\pi h})}&\frac{-\mathrm{i}e^{-\pi h+\pi t}(e^{\pi h}+e^{\pi t})(1+e^{2\pi t})}{(1+e^{\pi t})(e^{\pi h}+e^{2\pi t})(1+e^{\pi t-\pi h})}\\ \frac{\mathrm{i}e^{-\pi h+\pi t}(e^{2\pi h}+e^{2\pi t})(1+e^{\pi t})}{(1+e^{\pi t})(e^{\pi h}+e^{2\pi t})(1+e^{\pi t-\pi h})}&\frac{-(e^{\pi h}+e^{\pi t})(1+e^{2\pi t})}{(1+e^{\pi t})(e^{\pi h}+e^{2\pi t})(1+e^{\pi t-\pi h})}\end{pmatrix}.$$

Theorem 5. *A Fredholm operator d of the form (3) satisfies the inequalities (case V)*

$$|b_0^+(m_0\pm 0)|>|b_1^+(m_0\pm 0)|,\ \ |b_0^-(m_0\pm 0)|<|b_1^-(m_0\pm 0)| \qquad (18)$$

or the inequalities (case VI)

$$|b_0^-(m_0\pm 0)|>|b_1^-(m_0\pm 0)|,\ \ |b_0^+(m_0\pm 0)|<|b_1^+(m_0\pm 0)| \qquad (19)$$

iff there are exist non degenerate continuous matrix functions of order 2 w_1 and s_1 on $\overline{\mathbf{R}}_{m_0}$ such that the operator $e=w_1^{-1}d(m_0)s_1$ has the form (4) in case V or the form (5) in case VI for $l=1$. In the case V the matrices $w(m_0)(+\infty)$ and $s(m_0)(+\infty)$ are diagonal and the matrices $w(m_0)(-\infty)$ and $s(m_0)(-\infty)$ are skew diagonal while in the case VI we have the opposite situation.

Moreover there exists a homotopy in the class of Fredholm operators of the algebra $C^*(B,U_h)$ between the operator d and an operator d_1 and operators of zero indexes $w_2,s_2\in B$ such that in case V

$$d'=w_2d_1s_2(PU_h+Q)^{-1}\in B,$$

and in case VI

$$d'=w_2d_1s_2(P+QU_h)^{-1}\in B.$$

And

$$\mathrm{ind}\ d=\mathrm{ind}\ d'.$$

Proof. The proof of this theorem is also based on theorem 1. The inequalities (18) and (19) can be obtained from the cases (i) and (ii) of lemma 2 by taking the corresponding infinities.

In case VI we have

$$e(t) = w_1^{-1}(t) d(m_0)(t) s_1(t) = \left(\begin{array}{cc} e_{11}^0(t) & 0 \\ 0 & 1 \end{array} \right) + \left(\begin{array}{cc} 1 & 0 \\ 0 & e_{22}^1(t) \end{array} \right) T_h,$$

where $t \in \overline{\mathbf{R}}_{\mathbf{m_0}}$.

Let us define the homotopy

$$e'_\tau(t) = \left(\begin{array}{cc} (1-\tau) e_{11}^0(t) & 0 \\ 0 & 1 \end{array} \right) + \left(\begin{array}{cc} 1 & 0 \\ 0 & (1-\tau) e_{22}^1(t) \end{array} \right) T_h, \ \tau \in [0,1].$$

This homotopy defines the homotopy of the operator $d(m_0)(t)$:

$$d_\tau(m_0)(t) = w_1(t) e'_\tau(t) s_1^{-1}(t),$$

where we also have

$$e'_0(t) = e(t), \ d_0(m_0)(t) = d(m_0)(t),$$

$$e'_1(t) = \left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) + \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) T_h,$$

$$d_1(m_0)(t) = w_1(t) \left[\left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) + \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) T_h \right] s_1^{-1}(t).$$

The extension of the homotopy to the points $(m_0, \pm, +)$ and $(m_0, \pm, -)$ is defined in the following way

$$d_\tau(m_0, \pm, +) = (1-\tau) b_0(m_0, \pm, +) + b_1(m_0, \pm, +) T_h,$$

$$d_\tau(m_0, \pm, -) = b_0(m_0, \pm, -) + (1-\tau) b_1(m_0, \pm, -) T_h.$$

Therefore

$$d_0(m_0, \pm, +) = d(m_0, \pm, +) = b_0^+(m_0 \pm 0) + b_1^+(m_0 \pm 0) T_h,$$

$$d_0(m_0, \pm, -) = d(m_0, \pm, -) = b_0^-(m_0 \pm 0) + b_1^-(m_0 \pm 0) T_h,$$

$$d_1(m_0, \pm, +) = b_1^+(m_0 \pm 0) T_h, \quad d_1(m_0, \pm, -) = b_0^-(m_0 \pm 0).$$

On M_+ and M_- we define the homotopy on the following way

$$d_\tau^+(m) = \varphi_\tau(m) b_0^+(m) + b_1^+(m) a_h(m),$$

$$d_\tau^-(m) = b_0^-(m) + \varphi_\tau(m) b_1^-(m) a_h(m),$$

where the function φ_τ is defined in the same way as in the proof of theorem 2.

The operators $(d'_\tau)^{\pm}(m)$, $d'_\tau(m_0, +, \pm)$, $d'_\tau(m_0, -, \pm)$ and $d'_\tau(m_0)$ are invertible and for each $\tau \in [0, 1]$ they define the symbol of a certain Fredholm operator d'_τ and

$$\text{ind } d = \text{ind } d_0 = \text{ind } d_1.$$

For $t = \pm\infty$ we have the diagonal matrices $w_2(t) = \widetilde{w}_1(t) w_1^{-1}(t)$ and $s_2(t) = = s_1(t)\widetilde{s}_1^{-1}(t)$, (here the matrices $\widetilde{w}_1(t)$ and $\widetilde{s}_1(t)$ are that defined in remark 3). These matrices can be extended by means of non vanishing continuous functions on $M_\pm$ in such a way that the indexes of the corresponding operators w_2 and s_2 will be equal to zero (see the proof of lemma 1).

The operator $d' = w_2 d_1 s_2 (P + QU_h)^{-1}$ belongs to the algebra B and

$$\text{ind } d = \text{ind } d'.$$

Now the index can be calculated by means of formula (1).

To reduce case V to case VI it is enough to multiply the operator d by U_h^{-1}. The proof is complete. □

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